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DIFFEOMORPHISMS AND VOLUME-PRESERVING EMBEDDINGS OF NONCOMPACT MANIFOLDS

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ABSTRACT. The theorem of J. Moser that any two volume elements of equal total volume on a compact manifold are diffeomorphism-equivalent is extended to noncompact manifolds: A necessary and sufficient condition (equal total and same end behavior) is given for diffeomorphism equivalence of two volume forms on a noncompact manifold. Results on the existence of embeddings and immersions with the property of inducing a given volume form are also given. Generalizations to nonorientable manifolds and manifolds with boundary are discussed.

The topics of this paper are the action of the diffeomorphism group of a noncompact paracompact oriented manifold on the space of C^∞ volume forms on the manifold and the existence of volume-form-preserving embeddings of such manifolds into euclidean spaces. The results are essentially generalizations to the case of noncompact manifolds of a theorem of Moser [6] and a corollary of that theorem. The theorem is that if M is a compact connected oriented manifold and if ω and τ are C^∞ volume forms on M such that $\int_M \omega = \int_M \tau$ then there is a C^∞ diffeomorphism φ: $M \to M$ such that $\varphi^* \tau = \omega$. The corollary is that if ω is a C^∞ volume form on such a manifold and if there is a C^∞ embedding (immersion) of M into a particular euclidean space R^k then there is a C^∞ embedding (immersion) of M into that euclidean space with the property that the volume form of the Riemannian metric induced on M by the embedding (immersion) is ω. Such an embedding or immersion will be called volume-preserving hereafter. The corollary as stated follows from the theorem as follows: If E: $M \to R^k$ is an embedding (immersion) and ω_E = the volume form of the Riemannian metric induced by E then for some positive real number λ, $\int_M \omega_{\lambda E} = \int_M \omega$. If φ: $M \to M$ is a diffeomorphism such that $\varphi^* \omega_{\lambda E} = \omega$, the existence of which is guaranteed by

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Moser's theorem, then $\lambda E \circ \varphi$: $M \to R^k$ is the required embedding (immersion).

The result obtained for noncompact manifolds, analogous to the first result, is:

THEOREM 1. *If M is a noncompact connected oriented manifold and if ω and τ are volume forms on M with $\int_M \omega = \int_M \tau < +\infty$ and if each end of the manifold M has finite ω volume if it has finite τ volume and infinite ω volume if it has infinite τ volume, then there is a diffeomorphism φ: $M \to M$ such that $\varphi^* \omega = \tau$.*

The definition of finiteness and infiniteness of volume of an end will be given later as well as an example showing that the end condition cannot be omitted.

The generalization of the embedding-immersion result to the case of noncompact manifolds is the following result:

THEOREM 2. *If a noncompact connected oriented manifold M of dimension greater than 1 has a proper embedding (immersion) in a euclidean space R^k of dimension greater than the dimension of M, then for any volume form ω on M there is a proper embedding (immersion) of M in R^k which is volume-preserving for ω.*

Results corresponding to Theorems 1 and 2 hold for noncompact manifolds with boundary (see [1] and [2] for the case of compact manifolds with boundary). These results are discussed later in the paper, but for the sake of brevity they are omitted in this introduction.

In [4], it is shown that if ω and τ are real analytic volume forms on a real analytic noncompact oriented manifold M, then there is a real analytic diffeomorphism φ: $M \to M$ with $\varphi^* \omega = \tau$ if (and only if) there is a C^∞ diffeomorphism ψ: $M \to M$ such that $\psi^* \omega = \tau$. Thus the real analytic problem reduces to the C^∞ problem discussed in the present paper.

The hypothesis that M have a proper immersion in $\mathbf{R}^k$ ($k >$ dimension M) in Theorem 2 can be replaced by the hypothesis that M have an immersion in $\mathbf{R}^k$, not necessarily proper: the results of Hirsch [5] show that if any immersion exists then a proper immersion exists (here $k \ngtr$ dimension M is needed). Specifically, a C^∞ proper map always exists and by Theorem 5.10 of [5] any map can be uniformly approximated by an immersion provided some immersion exists. Since the uniform approximation of a proper map is proper, the conclusion follows (cf. the proof of Lemma 5 in §3).

The remainder of this paper is organized as follows: In §1, the concept of an end of a manifold with volume form having finite or infinite volume is discussed and an example given to show the role of the end condition in

Theorem 1. In §2, the proof of Theorem 1 is discussed briefly and in §3 the proof of Theorem 2 is discussed.

In §4, generalizations of Theorems 1 and 2 to the case of noncompact manifolds with boundary and to the case of odd forms on nonorientable manifolds are discussed.

Throughout the paper a manifold is taken to be connected by definition, unless otherwise specified.

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1. Ends of manifolds and the action of the diffeomorphism group on volume forms. Suppose M is a noncompact paracompact manifold. If K and K' are compact subsets of M with $K \subset K'$, then any component C' of $M - K'$ is contained in a unique component C of $M - K$. An *end* of M is an element of the inverse limit system $\{K, \text{components of } M - K\}$ indexed by $\{K | K$ compact subset of $M\}$ directed by inclusion as indicated.

Suppose M is orientable and has a volume form ω. Then an end ε of M has by direction *finite volume* if there is a compact set K such that the volume of the component of $M - K$ containing the end ε is finite. An end has by definition *infinite volume* if it does not have finite volume. Note that an end with finite volume does not have a numerical (finite) volume: The volume of the component of $M - K$ containing a finite-volume end depends on the choice of K; in fact, by a suitable choice of K, this volume can be made arbitrarily small.

To see the role of the end condition in Theorem 1, consider the case

$$M = R^2 - \{(0,0)\}, \quad \omega = dx \wedge dy, \quad \tau = (dx \wedge dy)/F(x^2 + y^2),$$

where $F\colon R^+ \to R^+$ is a C^∞ function such that

$$F(r) = 1 \quad \text{if } r > 1 \quad \text{and} \quad F(r) = r^2 \quad \text{if } 0 < r < \tfrac{1}{2}.$$

Clearly $\int_M \omega = \int_M \tau = +\infty$. However, the end condition is violated: $R^2 - \{(0, 0)\}$ has two ends, one identifiable with (0, 0), the other with the "point at infinity" in R^2. The "point at infinity" end has infinite τ volume and infinite ω volume as well. However, the (0, 0)-end has finite ω volume but infinite τ volume.

It is in fact the case that there is no diffeomorphism $\varphi\colon M \to M$ such that $\varphi^*\tau = \omega$. To see this observe that τ has the property that any embedded circle (S^1) in M which is not homologous to 0 divides M into two components, each having infinite τ-volume. Clearly this property is diffeomorphism invariant, i.e., the same property will hold for $\varphi^*\tau$ for any diffeomorphism φ: $M \to M$. But ω does not have this property: one component of $M - \{(x, y)|x^2 + y^2 = 1\}$ has finite ω-volume. Thus there is no diffeomorphism φ: $M \to M$ such that $\varphi^*\tau = \omega$.

It can happen that a diffeomorphism φ: $M \to M$ can exist with $\varphi^*\omega = \tau$ even if ω and τ do not satisfy the end condition. What is required is that there should be a diffeomorphism acting on the ends so as to make the end condition satisfied. Precisely, there is a diffeomorphism φ: $M \to M$ (for any noncompact paracompact M) such that $\varphi^*\omega = \tau$ if and only if $\int_M \omega = \int_M \tau < +\infty$ and there is a diffeomorphism φ_1: $M \to M$ such that $\varphi_1^*\omega$ and τ satisfy the end condition of Theorem 1, i.e., each end if M has its $\varphi_1^*\omega$ and τ volumes either both finite or both infinite. An example of this possibility (where φ_1 cannot be the identity) is obtained by again taking $M = R^2 - \{(0, 0)\}$ and taking

$$\omega = (dx \wedge dy)/(x^2 + y^2)^2 \quad \text{and} \quad \tau = (dx \wedge dy).$$

Here ω and τ have each one infinite and one finite volume but the finiteness occurs for different ends in the two cases. But if φ_1: $M \to M$ is the diffeomorphism $\varphi_1((x, y)) = (x, y)/(x^2 + y^2)$ then $\varphi_1^*\omega$ and τ both have the (0, 0) end finite volume and the "point at infinity" end infinite volume. This follows from observing that φ_1 interchanges the two ends. It is not always the case (for general M) that the diffeomorphism group acts transitively on the set of ends. Thus the existence of a suitable φ_1 is usually a more subtle question than the question of whether the cardinality of the set of finite-volume ends for ω equals the cardinality of the set of finite-volume ends for τ and similarly for infinite-volume ends.

2. The proof of Theorem 1. The following lemmas will be used in the proofs of both Theorem 1 and Theorem 2.

LEMMA 1. *If ω and τ are two volume forms on an oriented manifold M and if there is a connected compact set K with the properties that the support of $\omega - \tau$ is contained in the interior $\mathring{K}$ of K and that $\int_K \omega = \int_K \tau$, then there is a diffeomorphism φ: $M \to M$ such that $\varphi|(M - K)$ is the identity mapping and such that $\varphi^*\omega = \tau$.*

In Lemma 1, the support of $\omega - \tau$ is as usual the closure of the set $\{p \in M|(\omega - \tau)(p) \neq 0\}$. Lemma 1 is an immediate consequence of the argument used by Moser [6] to establish the theorem for compact manifolds referred to in the introduction of this paper.

LEMMA 2. *Suppose that M is an oriented manifold and that ω and τ are volume forms on M. Suppose also that N is a connected codimension 1* *submanifold of M and U is a tubular neighborhood of N (i.e. a neighborhood which is diffeomorphic to* $N \times (-1, 1)$ *under a diffeomorphism which takes N to* $N \times 0$). *Let* U_+ *and* U_- *be the components of* $U - N$. *Then there is a diffeomorphism* $\varphi: M \to M$ *such that* $\varphi|(M - U)$ *is the identity,* $\varphi^*\omega = \tau$ *on some neighborhood* $V \subset U$ *of N, and* $\int_{U_+}\varphi^*\omega = \int_{U_+}\omega$ *and* $\int_{U_-}\varphi^*\omega = \int_{U_-}\omega$.

PROOF OF LEMMA 2. Let W be a tubular neighborhood of N having compact closure $\overline{W} \subset U$. Let $F, G: M \to R$ be C^∞ functions with support in W and with the properties that $F = 1$, $G = 1$ in a neighborhood of N, $G < 1$ on M, and

$$\int_{U_+ \cap W} (1 - G)\omega + F\tau = \int_{U_+ \cap W} \omega$$

and

$$\int_{U_- \cap W} (1 - G)\omega + F\tau = \int_{U_- \cap W} \omega.$$

Since the support of $\omega - ((1 - G)\omega + F\tau)$ is contained in W and hence in the interior of the compact set $\overline{W}$ it follows from Lemma 1 that there is a diffeomorphism $\varphi: M \to M$ with $\varphi|(M - \overline{W}) =$ identity and $\varphi^*\omega = (1 - G)\omega + F\tau$. □

LEMMA 3. *Suppose M is a noncompact orientable manifold of dimension n and* $\{K_i | i = 1, 2, \dots\}$ *is a sequence of n-dimensional compact connected submanifolds-with-boundary such that* $\bigcup_{i=1}^{+\infty} K_i = M$ *and* $K_i \cap K_j$ *for all* i, j, $i \neq j$, *is either empty or is an* $(n - 1)$*-dimensional submanifold of M which is contained in the boundary of* K_i *and also in the boundary of* K_j. *Suppose also that* ω *and* τ *are volume forms on M such that* $\int_{K_i} \omega = \int_{K_i} \tau$ *for each* $i = 1, 2, \cdots$. *Then there is a diffeomorphism* $\varphi: M \to M$ *such that* $\varphi^*\omega = \tau$.

PROOF OF LEMMA 3. According to Lemma 2 there is a diffeomorphism $\psi: M \to M$ which equals the identity outside the union of disjoint tubular neighborhoods of the connected components of the boundaries of the K_i with the properties that $\psi^*\omega = \tau$ near the union of the boundary components and $\int_{K_i} \psi^*\omega = \int_{K_i} \tau$ for all $i = 1, 2, \dots$. (The last condition here follows from the final two equalities in the conclusion of Lemma 2: the total volume on either side of N is unchanged by the diffeomorphism φ there.) By Lemma 1 there is, for each $i = 1, 2, \dots$, a diffeomorphism $\varphi_1: K_i \to K_i$ such that $\varphi_i =$ identity in a neighborhood of the boundary of K_i and $\varphi_i^*(\psi^*\omega) = \tau$ on K_i. Clearly there is a diffeomorphism $\varphi_\infty: M \to M$ such that $\varphi_\infty | K_i = \varphi_i$. Then $(\psi\varphi_\infty)^*\omega = \tau$. □

For the proof of Theorem 1, one more lemma is needed:

LEMMA 4. *Let K be a compact subset of a noncompact oriented manifold M with volume form ω. Then a connected component C of $M - K$ has infinite volume if and only if C contains an end of K that has infinite volume.*

PROOF OF LEMMA 4. Clearly if C contains an end of infinite volume then C has infinite volume. To prove the converse, suppose C has infinite volume. Let $\{K_i | i = 1, 2, \dots\}$ be a sequence of compact subsets of M having the properties that $K_1 = K$, $\bigcup K_i = M$, $K_i \subset \mathring{K}_{i+1}$ for all $i = 1, 2, \dots$ and for all $i = 2, 3, \dots$, $M - K_i$ has only finitely many components all of which have noncompact closures. (Such a sequence is easily constructed.) Now consider the components of $M - K_2$ which lie in C. Clearly one of these components has infinite volume for otherwise the facts that there are only finitely many of these components and that $K_2 \cap C$ has compact closure would imply that C has finite volume. Let C_2 be one of the components of $M - K_2$ which has infinite volume. Repeating the argument one obtains a component C_3 of $M - K_3$ which is contained in C_2 and which has infinite volume, etc. The sequence $C \supset C_2 \supset C_3 \supset C_4 \supset \cdots$ of components of $M - K$, $M - K_2$, $M - K_3$, $M - K_4, \dots$ gives an end with infinite volume. □

PROOF OF THEOREM 1. Choose a C^∞ function $F: M \to R$ such that for all $\alpha \in R$, $F^{-1}((-\infty, \alpha])$ is compact and such that each positive integer is a noncritical value of F. Define a sequence of compact subsets of M as follows: Let p be a point of $F^{-1}((-\infty, +1])$, which is assumed without loss of generality to be nonempty. Put M_1 = the union of the connected component C of $F^{-1}((-\infty, +1])$ which contains p with the components of $M - C$ which have compact closure. Put M_2 = the connected component of $F^{-1}((-\infty, +2])$ which contains M_1 union the closure-compact components of the complement of that component, etc. Clearly, for each $i = 1, 2, \dots, M_i$ is connected; also for each $i = 1, 2, \dots$, $M_i \subset \mathring{M}_{i+1}$; and $\bigcup_{i=1}^{+\infty} M_i = M$. Furthermore, for each $i = 1, 2, \dots$, every component of $M - M_i$ has noncompact closure.

The proof will consist of an iterated construction of volume forms ω_1 and τ_1, ω_2 and τ_2, etc. and diffeomorphisms $\varphi_1: M \to M$ and $\psi_1: M \to M$, φ_2 and ψ_2, etc. such that $\varphi_1^*\omega = \omega_1$ and $\psi_1^*\tau = \tau$, $\varphi_2^*\omega_1 = \omega_2$ and $\psi_2^*\tau_1 = \tau_2$, etc. This iterative construction is as follows:

Set $i_1 = 1$. Choose ω_1 and τ_1 volume forms on M with the following properties:

(a) the supports of $\omega - \omega_1$ and $\tau - \tau_1$ are compact;

(b) $\int_{M_{i_1}} \omega_1 = \int_{M_{i_1}} \tau_1 = \max(\int_{M_{i_1}} \omega, \int_{M_{i_1}} \tau)$;

(c) $\int_{\text{supp}(\omega - \omega_1)} \omega_1 = \int_{\text{supp}(\omega-\omega_1)} \omega$ and $\int_{\text{supp}(\tau-\tau_1)} \tau_1 = \int_{\text{supp}(\tau-\tau_1)} \tau$ where $\text{supp}(\omega - \omega_1)$ = support of $(\omega - \omega_1)$ and $\text{supp}(\tau - \tau_1)$ = support of $(\tau - \tau_1)$;

(d) $\int_C \omega_1 = \int_C \tau_1$ for each component C of $M - M_{i_1}$;

(e) for each component C of $M - M_{i_1}$, $C - (\text{supp}(\omega - \omega_1) \cup \text{supp}(\tau - \tau_1))$ is connected.

That such choices are possible is clear since the only condition involving nonfinite volumes is (d), and the possibility of that condition being satisfied is guaranteed by the end condition hypothesis combined with Lemma 4.

Informally, the ω_1 and τ_1 are ω and τ with some volume moved around so that ω_1 and τ_1 have equal total on $M_{i_1} = M_1$ and so that the components of $M - M_{i_1}$ have equal totals. Note that if the end condition hypothesis were omitted then this equal totals on the components of $M - M_{i_1}$ might not be arrangeable consistently with condition (a).

Now there exist by Lemma 1 diffeomorphisms $\varphi_1: M \to M$ and $\psi_1: M \to M$ such that $\varphi_1^*\omega = \omega_1$ and $\psi_1^*\tau = \tau_1$ and such that φ_1 and ψ_1 = identity outside arbitrarily small neighborhoods U_1 and V_1 of $\text{supp}(\omega - \omega_1)$ and $\text{supp}(\tau - \tau_1)$. In particular, these neighborhoods U_1 and V_1 can be taken so that they have compact closure and so that the union of their closures does not disconnect any component of $M - M_{i_1}$.

For the second stage of the construction, pick a positive integer $i_2 > i_1 + 2$ and so large that φ_1 and ψ_1 = the identity on M_{i_2-2}. Choose volume forms ω_2 and τ_2 such that:

(a) $\text{supp}(\omega_1 - \omega_2)$ and $\text{supp}(\tau_1 - \tau_2)$ are compact; $\text{supp}(\omega_1 - \omega_2) \cap M_{i_1}$ is empty; and $\text{supp}(\tau_1 - \tau_2) \cap M_{i_1}$ is empty; and $\text{supp}(\omega_1 - \omega_2) \cap \overline{U}_1$ is empty and $\text{supp}(\tau_1 - \tau_2) \cap \overline{V}_1$ is empty;

(b) $\int_C \omega_2 = \int_C \tau_2 = \max(\int_C \omega_1, \int_C \tau_1)$ for each component C of $M_{i_2} - M_{i_1}$;

(c) $\int \omega_1 = \int \omega_2$ over $\text{supp}(\omega_1 - \omega_2)$, $\int \tau_1 = \int \tau_2$ over $\text{supp}(\tau_1 - \tau_2)$;

(d) $\int_C \omega_2 = \int_C \tau_2$ for each component C of $M - M_{i_2}$;

(e) for each component C of $M - M_{i_2}$, $C - (\text{supp}(\omega_1 - \omega_2) \cup \text{supp}(\tau_1 - \tau_2))$ is connected.

Again one then obtains diffeomorphisms $\varphi_2: M \to M$, $\psi_2: M \to M$ such that $\varphi_2\omega_1 = \omega_2$ and $\psi_2\tau_1 = \tau_2$. These diffeomorphisms can and are to be taken to be equal to the identity outside neighborhoods U_2 and V_2 of $\text{supp}(\omega_1 - \omega_2)$ and $\text{supp}(\tau_1 - \tau_2)$, respectively, which are so small that their closures are disjoint respectively from $\overline{U}_1$ and $\overline{V}_1$ and so that $M_{i_1} \cap U_2$ is empty and $M_{i_1} \cap V_2$ is empty. Also, the neighborhoods U_2 and V_2 are to be chosen so that for each component C of $M - M_{i_2}$, C is not disconnected by the union of $\overline{U}_2$ and $\overline{V}_2$.

The construction now continues inductively. The φ_i and ψ_i were chosen so that there are diffeomorphisms $\varphi_\infty: M \to M$ and $\psi_\infty: M \to M$ such that $\varphi_\infty | U_i = \varphi_i$ and $\psi_\infty | V_i = \psi_i$ for all i and $\varphi_\infty|(M - \bigcup_{i=1}^\infty U_i)$ = identity and $\psi_\infty|(M - \bigcup_{i=1}^\infty V_i)$ = identity. Now the volume forms $\varphi_\infty^*\omega$ and $\psi_\infty^*\tau$ have equal total volumes on M_{i_1} and on each component of $M_{i_k} - M_{i_{k-1}}$, $k = 2, 3, 4, \ldots$. Lemma 3 implies that there is a diffeomorphism $\eta: M \to M$ such that $\eta^*(\psi^*_\infty \tau) = \varphi^*_\infty \omega$. Thus $\tau = (\varphi\eta^{-1}_\infty \psi^{-1}_\infty)^*\omega$. □

3. The proof of Theorem 2. The proof of Theorem 2 will be given in this section. Note that Theorem 2 does not follow from Theorem 1 as the embedding (immersion) result for compact manifolds follows from Moser's theorem: the end condition in Theorem 1 and also the fact that an infinite volume embedding or immersion cannot be converted to a finite one by multiplication by a constant necessitate a separate argument.

The proof of Theorem 2 depends upon a preliminary result about proper embeddings and immersions in euclidean space.

LEMMA 5. *If M is a noncompact oriented manifold of dimension* > 1, *if there is a proper embedding* (*immersion*) *of M in a euclidean space R^k, $k >$* $\dim M + 1$, *and if ω is a volume form on M then there is a proper embedding* (*immersion*) *E: $M \to R^k$ such that $\omega_E < \omega$, where ω_E is the volume form of the Riemannian metric induced by E.*

PROOF OF LEMMA 5. First, suppose that there is a proper embedding of M in R^k; it will be shown that there is a proper embedding E_1: $M \to R^k$ such that $E_1(M)$ is contained in an open half-space of R^k. For this purpose, the cases $k > 3$ and $k = 3$ need to be considered separately:

If $k > 3$, then any proper semi-infinite embedded curve C: $[0, +\infty) \to R^k$ is unknotted, so for instance there is a diffeomorphism φ: $R^k \to R^k$ such that $\varphi \circ C$: $[0, +\infty) \to R^k$ is the map $(\varphi \circ C)(t) = (t, 0, \dots, 0)$. Since it is easy to see that there is a proper semi-infinite embedded curve in $R^k - E(M)$, it follows that there is a proper embedding E_0: $M \to R^k$ such that $E_0(M) \cap \{(t, 0, \dots, 0) | t > 0\}$ is empty, namely $\varphi \circ E$. Since E_0 is proper, there is a tube-shaped neighborhood U of $\{(t, 0, \dots, 0)|t > 0\}$ such that $\overline{U} \cap E_0(M)$ is empty. But there is a diffeomorphism φ_1: $R^k \to R^k$ such that $\varphi_1(U)$ is an open half-space. Then $\varphi_1 \circ E_0$ is a proper embedding with $(\varphi_1 \circ E_0)(M)$ contained in the open half-space $R^k - \varphi_1(\overline{U})$. If $k = 3$, then the dimension of $M = 2$. Let F: $M \to R$ be a function with the properties that $F^{-1}((-\infty, \alpha])$ is compact for all $\alpha \in R$, each $i = 1, 2, \dots$ is a noncritical value of F, and $F(p) > 1/2$ for all $p \in M$. Each of the sets $M_i = F^{-1}([i - 1, i])$, $i = 1, 2, \dots$, is a compact 2-dimensional oriented manifold with boundary, i.e. a finite union of spheres with a certain number of handles attached and a certain number of tubes-with-boundary attached. It is easy to see that each M_i can be embedded between $\{(x, y, i - 1)|x, y \in R\}$ and $\{(x, y, i)|x, y \in R\}$ in such a way that the boundary components of M_i on which $F = i - 1$ are embedded in $\{(x, y, i - 1)\}$ and those on which $F = i$ are embedded in $\{(x, y, i)\}$. Clearly, also, these embeddings can be chosen so that they fit together to form an embedding of M itself which lies in the half-space $\{(x, y, z)|z > 0\}$.

Let now $E_1: M \to R^k$ be a proper embedding such that $E_1(M)$ is contained in a half-space H. Choose a diffeomorphism $\varphi_2: R^k \to R^k$ which maps H into a subset $\{(x_1, x_2, \ldots, x_n)|x_n > 0, \Sigma_{i=1}^{n-1} x_i^2 < 1\}$; then $\varphi_2 \circ E_1: M \to R^k$ is a proper embedding with

$$(\varphi_2 \circ E_1)(M) \subset \left\{(x_1, x_2, \ldots, x_n)|x_n > 0, \sum_{i=1}^{n-1} x_i^2 < 1\right\}.$$

Now define a diffeomorphism $\varphi_3: R^k \to R^k$ by

$$(x_1, \ldots, x_{n-1}, x_n) \to (F(x_n)x_1, \ldots, F(x_n)x_{n-1}, x_n)$$

where F is a positive C^∞ function with $F(x_n) < 1$ for all x_n. Note that, because the dimension of M is at least 2, if $\tau' =$ the volume form induced by $\varphi_2 \circ E_1$ and $\tau'' =$ the volume form induced by $\varphi_3 \circ \varphi_2 \circ E_1$ then

$$\tau''(p) < F(z((\varphi_2 \circ E_1)(p)))\tau'(p).$$

Since $\varphi_2 \circ E_1$ is proper, it follows that F can be chosen so that $\tau'' < \omega$.

The arguments just given for the case of embeddings hold mutatis mutandis for the case of immersions with one exception: if $E: M^n \to \mathbf{R}^{n+1}$ is a proper immersion then $\mathbf{R}^{n+1} - F(M)$ may fail to contain a proper image of $[0, +\infty)$. To avoid this difficulty, one can construct the required proper immersion image in a half-space by choosing a transversal hyperplane, reflecting the part of the original immersion that goes into a fixed half-space across the hyperplane and smoothing along the resulting "fold".

Alternatively, one can treat the immersion case by applying the general theory of immersions of Hirsch [5] as follows: Let $F: M \to \mathbf{R}$ be a C^∞ function with $F^{-1}((-\infty, \alpha])$ compact for all $\alpha \in \mathbf{R}$ (as before). The map F_1: $M \to \mathbf{R}^N$ defined by

$$F_1(p) = (0, \ldots, 0, F(p)), \qquad p \in M,$$

is proper. By Theorem 5.10 of [5], the existence of any immersion (proper or not, in fact) $E: M \to \mathbf{R}^N$ implies (since $N > n$) that there is an immersion E_1: $M \to \mathbf{R}^N$ which approximates F_1, say $\|F_1(p) - E_1(p)\| < 1$ for all $p \in M$. Then E_1 is proper and $E_1(M)$ is contained in the half-space $\{(x_1, \ldots, x_N)|x_N > 1 + \min_M F\}$. The rest of the proof is as in the embedding case. □

Proof of Theorem 2. Let $E_0: M \to R^k$ be a proper embedding (immersion) of M in R^k such that $\omega_{E_0} < \omega$: the existence of E_0 is guaranteed by Lemma 5. Choose a function $F: M \to R$ as in previous proofs (i.e. F with $F > 1/2, i = 1, 2, \ldots,$ noncritical values, and with $F^{-1}((-\infty, \alpha])$ compact for all $\alpha \in R$). Then define a sequence M_i, $i = 1, 2, \ldots,$ by $M_i = F^{-1}([i - 1, i])$, each M_i being a compact submanifold with boundary. Let M_j', $j = 1, 2, \ldots,$ be the compact connected submanifolds with boundary which are the components of the M_i's. According to Lemma 3, the theorem will follow if there exists an embedding E_1: $M \to R^k$ such that for each j, $\int_{M_j'} \omega_{E_1} = \int_{M_j'} \omega$: For then there is a diffeomorphism φ: $M \to M$ such that $\varphi^* \omega_{E_1} = \omega$ and thus $\omega_{E_1 \circ \varphi} = \omega$ and $E_1 \circ \varphi$ is the required volume-preserving embedding (immersion). Now $\int_{M_j'} \omega_{E_0} < \int_{M_j'} \omega$ so what is needed is a perturbation of E_0 which increases the induced volume. Such perturbations which preserve the property of being a proper embedding (immersion) can be constructed using a technique developed in [7].

Specifically, the construction is as follows: The global perturbation is obtained from a sequence of local ones. For each $j = 1, 2, \ldots,$ let p_j be a point of $\mathring{M}_j'$; a perturbation of the embedding (immersion) in a neighborhood of each p_j will be found so that the required conditions are satisfied. For this purpose, for each $j = 1, 2, \ldots,$ let N_j be a unit vector normal at p_j to the image of M. Choose a neighborhood U_j of p_j in M such that $\overline{U}_j \subset \mathring{M}_j'$ and such that the composition of E_0 with the orthogonal projection on the orthogonal complement of N_j is an embedding of $\overline{U}_j$. Then for any C^∞ function F_j: $M \to R$ with supp $F_j \subset U_j$, and $|F_j| < 1$, the mapping $E_0 + F_j N_j$ defined by

$$\Big(E_0 + \sum_j F_j N_j\Big)(p) = E_0(p) + \sum_j F_j(p) N_j$$

is a proper immersion (note that the vector sum is finite, having in fact at most two nonzero terms). Furthermore, if E_0 is an embedding, there is a sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ of positive numbers such that if $|F_j| < \varepsilon_j$ then $E_0 + \Sigma_j F_j N_j$ is an embedding. On the other hand, by choosing the F_j's to oscillate rapidly while still satisfying $|F_j| < 1$ (and $|F_j| < \varepsilon_j$ in the embedding case) it can clearly be arranged that

$$\int_{M_j'} \omega_{E_0 + \Sigma_j F_j N_j} > \int_{M_j'} \omega.$$

Consider $\omega_{E_0 + \Sigma_j \lambda_j F_j N_j}$ where $0 < \lambda_j < 1$. Clearly the integral of this volume form over M_j' is a continuous function of λ_j (and is independent of λ_k, $k \neq j$). On the other hand, if $\lambda_j = 1$ then the integral over M_j' is $> \int_{M_j'} \omega$ while if $\lambda_j = 0$, the integral is $< \int_{M_j'} \omega$. Hence there exist $\lambda_j, j = 1, 2, \ldots,$ $0 < \lambda_j < 1$, such that for all $j = 1, 2, \ldots$

$$\int_{M_j'} \omega_{E_0 + \Sigma_j \lambda_j F_j N_j} = \int_{M_j'} \omega$$

as required. □

4. Generalizations to noncompact manifolds with boundary and nonorientable manifolds. The results of the previous sections have generalizations to manifolds with boundary and to odd forms on nonorientable manifolds (possibly with boundary). These generalizations and their proof are formally virtually identical to the theorems as given, and the detailed formulations and proof of these results will be left to the reader. A few comments are in order, however:

In the (orientable) manifold with boundary case, in attempting to transform one volume form ω to another τ via a diffeomorphism, one can first find a diffeomorphism φ_1: $M \to M$ fixing the boundary ∂M of M pointwise and equal to the identity except near ∂M such that $\varphi_1^*\omega = \tau$ near ∂M. Then proceeding as before (provided the appropriate total volume and end conditions are satisfied) one need only consider diffeomorphisms with support away from ∂M, and one finds φ: $M \to M$ such that $\varphi^*\omega = \tau$ and $\varphi|\partial M =$ identity. To obtain φ_1 choose a tubular neighborhood $U \simeq \partial M \times [0, 1)$ of M and let $\varphi_1((p, t)) = (p, f(p, t))$ where $f(p, 0) = 0$ and $\partial f/\partial t = \tau(p, t)/\omega(p, t)$. This gives φ_1 near ∂M and φ_1 is extended globally by an obvious procedure.

To turn to the nonorientable case, recall that if M is nonorientable then a nowhere vanishing odd form ω on M is by definition a volume form $\tilde{\omega}$ on $\tilde{M}$, the orientable double cover of M, such that $I^*\tilde{\omega} = -\tilde{\omega}$, where I is the nontrivial covering transformation (with I^2 identity) on $\tilde{M}$. To prove the desired generalizations to this case, one can proceed as follows: Choose a connected smoothly bounded open set D in $\tilde{M}$ such that $D \cap I(D) = \partial D$ and $D \cup I(D) = M$. Then carry out the constructions of the previous sections on D and extend to $I(D)$ by applying I. The result clearly has the required I-equivariance. The only delicate point is to be sure of smoothness across ∂D. For this point, one must choose an equivariant tubular neighborhood of ∂D and use this to make the constructions operating near ∂D I-equivariant. The existence of a fundamental domain D as described follows, as kindly pointed out to the authors by W. Meeks and B. O'Neill, from considering the classifying map of the $\mathbf{Z}_2$ orientation bundle $\tilde{M} \to M$, say G: $M \to \mathbf{R}P^N$, N large. If G is taken transversal to a hyperplane P in $\mathbf{R}P^N$, then $G^{-1}(P)$ is a smooth hypersurface and $M - G^{-1}(P)$ is orientable. There is an open subset D of M which is maximal among those connected open subsets of M which are orientable and the boundary of which is a union of components of $G^{-1}(P)$. Such a D has the required properties.

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